Math 1C Midterm 2
Fall 2012
Riverside City College

Instructions: All work is to be shown, done in pencil, legible, simplified and answers/solutions are to be boxed in the space provided. You are to work alone and any student caught cheating will receive a zero. You are allowed to only use a scientific calculator! Answers/solutions to word problems are to be written in a complete sentence with the correct units. Moreover, you will be allotted 2 hours for this exam. If you finish early, then go back and check your work and answers. Students are not allowed to leave the room and return. So, if you have to use the restroom, please use it before starting the exam. Before you begin, remember to turn all phones off. Failure to comply with any of these instructions may result in a zero.
1. (4pts) Describe the level curves of \( z = \sqrt{1-x^2-4y^2} \). Sketch these curves when \( z = 0, \frac{1}{4}, \frac{1}{2}, 1 \).

Let \( z = k \); \( k \geq 0 \)
\[
\Rightarrow \sqrt{1-x^2-4y^2} = k
\]
\[
\Rightarrow 1-x^2-4y^2 = k^2
\]
\[
\Rightarrow x^2 + 4y^2 = 1-k^2
\]
\[
\Rightarrow \frac{x^2}{(1-k^2)} + \frac{y^2}{\frac{1}{4}(1-k^2)} = 1
\]

Clearly, the level curves are ellipses.

When \( z = 0 \)
\[
\Rightarrow x^2 + 4y^2 = 1
\]
\[
\Rightarrow \frac{16x^2}{15} + \frac{64y^2}{15} = 1
\]

When \( z = \frac{1}{4} \)
\[
\Rightarrow x^2 + 4y^2 = \frac{15}{16}
\]
\[
\Rightarrow \frac{4x^2}{3} + \frac{16y^2}{3} = 1
\]

When \( z = \frac{1}{2} \)
\[
\Rightarrow x^2 + 4y^2 = \frac{3}{4}
\]
\[
\Rightarrow \frac{x^2}{\frac{3}{4}} + \frac{16y^2}{\frac{3}{4}} = 1
\]

When \( z = 1 \)
\[
\Rightarrow x^2 + 4y^2 = 0
\]
Only \((0,0)\) is true.

\[\text{Note: The "largest" ellipse is when } z = 0 \text{ and gets "smaller" when } z \text{ approaches } 1.\]
2. (4pts) Show that the following limit does not exist

$$\lim_{(x,y) \to (0,0)} \frac{x^2 - 2y}{y^2 + 2x}.$$ 

\(C_1\): Along the x-axis: \((x,0)\)

$$\lim_{(x,0) \to (0,0)} \frac{x^2}{x} = 0 = L_1$$

\(C_2\): Along the y=x

$$\lim_{(x,x) \to (0,0)} \frac{x^2 - 2x}{x^2 + 2x} = \lim_{(x,x) \to (0,0)} \frac{x - 2}{x + 2} = -1 = L_2$$

\[\therefore\text{ The limit does not exist because } L_1 \neq L_2.\]
3. (4pts) Use the epsilon-delta definition of a limit to prove \( \lim_{(x,y) \to (1,2)} (5x - 2y) = 1 \).

Let \( \epsilon > 0 \) be given. Choose \( \delta = \sqrt{\frac{\epsilon}{7}} > 0 \) so that if \( \sqrt{(x-1)^2 + (y-2)^2} < \delta \)

then \( |5x - 2y - 1| = |5x - 5 - 2y + 4| \)

\[ = |5(x-1) - 2(y-2)| \]

\[ \leq |5(x-1)| + |2(y-2)| \]

\[ \leq 5|x-1| + 2|y-2| \quad (\because \text{Triangle Inequality}) \]

\[ = 5\sqrt{(x-1)^2} + 2\sqrt{(y-2)^2} \]

\[ \leq 5\delta + 2\delta \quad (\because \text{Assumptions}) \]

\[ = 7\delta \]

\[ = \epsilon \quad \square \]
4. (4pts) Find the equation of the tangent line to the surface $x^2 + 4y^2 + 4z^2 = 9$ at the point $(1,1,1)$ that lies in the plane $y = 1$.

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When $y = 1$  \Rightarrow  x^2 + 4z^2 = 5
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Now, we need to find the slope at $(1,1)$ in the $xz$-plane when $y = 1$.

$\Rightarrow 2x + 8z \frac{dz}{dx} = 0$

$\Rightarrow \frac{dz}{dx} = -\frac{x}{4z}$

$m = \left. \frac{dz}{dx} \right|_{(1,1)} = -\frac{1}{4}$

So, $z - z_1 = -\frac{1}{4}(x - x_1)$

$\Rightarrow z - 1 = \frac{x - 1}{-4}$

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\text{tangent:}  \quad y = 1,  \quad \frac{z-4}{-4} = \frac{z-1}{1}
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or

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\text{tangent:}  \quad x = -4t + 4,  \quad y = 1,  \quad z = t + 1
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or

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\text{tangent:}  \quad \langle x, y, z \rangle = \langle 1, 1, 1 \rangle + t \langle -4, 1, 0 \rangle
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5. (4pts) The radius of a right circular cone is increasing at a rate of 7 in/min, while its height is decreasing at a rate of 20 in/min. Is the volume of the cone increasing or decreasing when \( r = 45 \) in and \( h = 100 \) in? How fast is the volume change then? 

(Hint: \( V = \frac{1}{3} \pi r^2 h \))

\[
\frac{dv}{dt} = \frac{2}{3} \pi r h \frac{dr}{dt} + \frac{1}{3} \pi r^2 \frac{dh}{dt}
\]

\[
\frac{dv}{dt} = \left( \frac{2}{3} \pi \cdot 45 \cdot 100 \right)(7) + \left( \frac{1}{3} \pi \cdot 45^2 \right)(-20)
\]

\[
\frac{dv}{dt} = 21000 \pi - 13500 \pi
\]

\[
\frac{dv}{dt} = 7500 \pi \text{ in}^3/\text{min}
\]

\[\text{The volume is increasing at a rate of } 7500 \pi \text{ in}^3/\text{min}.\]
6. (4pts) Are there any points on the hyperboloid \( x^2 - y^2 - z^2 = 1 \) where the tangent plane is parallel to the plane \( z = x + y \)? If possible, find the point(s). If not possible, explain why.

Let's find the eqn. of the tangent plane of \( x^2 - y^2 - z^2 = 1 \)

\[ \nabla f = \langle 2x, -2y, -2z \rangle \parallel \langle x, -y, -z \rangle \]

Clearly, \( x \neq 0 \) s.t. \( \langle x, -y, -z \rangle = \alpha \langle 1, 1, -1 \rangle \)

So there does not exist any points on \( x^2 - y^2 - z^2 = 1 \) s.t. its tangent plane is parallel to \( z = x + y \).
7. (4pts) Determine the nature of the critical points of the following function

\[ f(x, y) = x^2 - xy + y^2 + 2x + 2y. \]

\[
\begin{align*}
f_x &= 2x - y + 2 \\

f_y &= -x + 2y + 2 \\

\Delta &= (2x - y + 2) (x + 2y + 2) = 0 \\

4x - 2y &= -4 \\
-x + 2y &= -2 \\
3x &= -6 \\
x &= -2 \\
y &= -2
\end{align*}
\]

So, CP: (-2, 2)

Apply second derivative test!

\[
\begin{align*}
f_{xx} &= 2 \\
f_{yy} &= 2 \\
f_{xy} &= -1 \\
D &= (2)(2) - (-1)^2 > 0
\end{align*}
\]

\[ f(-2, -2) = -4 \] is a local min. value.
8. (4pts) Use the technique of the Lagrange Multiplier to find the maximum and minimum values of $x^2 + y^2 + z^2$ subject to the condition $z^2 = x^2 - 1$.

Let $f(x, y, z) = x^2 + y^2 + z^2$ and $g(x, y, z) = x^2 - z^2 - 1$.

$\Rightarrow \nabla f = \lambda \nabla g$

$\Rightarrow \langle 2x, 2y, 2z \rangle = \lambda \langle 2x, 0, -2z \rangle$

1. $2x = \lambda (2x)$
2. $2y = \lambda (0)$
3. $2z = \lambda (-2z)$
4. $z^2 = x^2 - 1$

\begin{align*}
\text{(1)} & \quad 2x(1 - \lambda) = 0 \\
& \quad x / 0 \lor \lambda / 1 \\
& \quad \text{(neither case makes sense!)}
\end{align*}

\begin{align*}
\text{(3)} & \quad 2z(1 + \lambda) = 0 \\
& \quad z = 0 \lor \lambda / -1
\end{align*}

\begin{align*}
& \text{if } z = 0 \Rightarrow x = \pm 1 \\
& \quad \phi(\pm 1, 0, 0) = 1
\end{align*}

\[ \phi(\pm 1, 0, 0) = 1 \text{ is the min. value} \]